

On the Measure of the Midpoints of the Cantor Set in \mathbb{R}

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Abstract

In this paper, we are going to discuss the following problem: Let T be a fixed set in \mathbb{R}^n , and let S and B be two subsets in \mathbb{R}^n such that for any x in S , there exists an r such that $x + rT$ is a subset of B . How small can B be if we know the size of S ? Stein proved that for n greater than or equal to 3 and T is a sphere centered at origin, then S having positive measure implies B has positive measure by using spherical maximal operator. Later, Bourgain and Marstrand proved the similar result for $n = 2$. Here we will show an example for why the result fails for $n = 1$.

1 Introduction

The problem in the abstract is included in the paper by Tamás Keleti [3]. The purpose of this paper is to construct a counterexample in \mathbb{R} for the following theorem which holds for dimensions greater than 1.

Theorem 1.1. *Let $S \subset \mathbb{R}^n$ ($n \geq 2$) be a set of positive Lebesgue measure. If $B \subset \mathbb{R}^n$ contains a sphere centered at every point of S , then B has positive Lebesgue measure.*

In 1976, Elias Stein [4] proved the case for when $n \geq 3$. Then in 1987 and 1986, Marstrand [2] and Bourgain [1] independently proved the theorem for dimension $n = 2$. However, the theorem is not true for $n = 1$.

Idea : To construct an counterexample, the idea is to start with a set B with zero length, and then construct a set S with positive measure made up of all midpoints from B . Notice that if S contains all the midpoints from B , then for any point $z \in S$, it will be the midpoint of a pair of points x, y in B . And $\{x, y\}$ will be a sphere centered at $z \in S$.

Definition 1.2 (1/3-Cantor set). *Let $C_1 = [0, 1] \setminus (\frac{1}{3}, \frac{2}{3})$, and*

$$C_i = \frac{1}{3}C_{i-1} \cup \left(\frac{2}{3} + \frac{1}{3}C_{i-1}\right), \quad \forall i \geq 2.$$

We define the 1/3-Cantor set to be $\mathcal{C} = \bigcap_{i=1}^{\infty} C_i$.

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The $1/3$ -Cantor set \mathcal{C} will take place of B in the theorem, and the midpoints from \mathcal{C} , denoted as $M_{\mathcal{C}}$, will take place of S . We define the **midpoint set** of A in \mathbb{R} as

$$M_A := \left\{ \frac{x+y}{2} : x, y \in A, x \neq y \right\}.$$

Since $\mathcal{C} = \cap_{i=1}^{\infty} C_i$, to get $\mathcal{L}^1(M_{\mathcal{C}}) > 0$ we will show that $M_{C_i} = (0, 1)$ for all i . After that, we then prove

$$M_{\mathcal{C}} = M_{\cap_{n=1}^{\infty} C_n} = \bigcap_{n=1}^{\infty} M_{C_n} = (0, 1).$$

Therefore, by setting $B = \mathcal{C}$ and $S = M_{\mathcal{C}}$, we will have $\mathcal{L}(S) = 1$ while $\mathcal{L}(B) = 0$, which will be our counterexample.

As a matter of fact, with the same construction, we may make $M_{\mathcal{C}}$ as large as we want by making copies of \mathcal{C} in other intervals!

2 A counterexample in \mathbb{R}

Theorem 2.1. *There exist $S, B \subset \mathbb{R}$ and B contains a sphere (in \mathbb{R}) centered at every point of S such that $\mathcal{L}(S) > 0$ but $\mathcal{L}(B) = 0$.*

In subsection 2.1, we will go over a few definitions and a couple of lemmas which we will use for the proofs of Theorem 2.1. We say proofs, because we will go over two distinct proofs; in subsection 2.2 we will go over an analytic induction proof and in subsection 2.3 we will provide a more geometric argument.

2.1 Definitions and lemmas

Remark 2.2. *In \mathbb{R} , a sphere of radius r centered at a point x will be two points $x - r$ and $x + r$.*

Definition 2.3 (i th Cantor partition set for $[0, 1]$). *Define*

$$P_i = \left\{ P_{C_i}^k : P_{C_i}^k = \left(\frac{k-1}{3^i}, \frac{k}{3^i} \right), k \in \{1, 2, 3, \dots, 3^i\} \right\}$$

to be the i th Cantor partition set of $(0, 1)$.

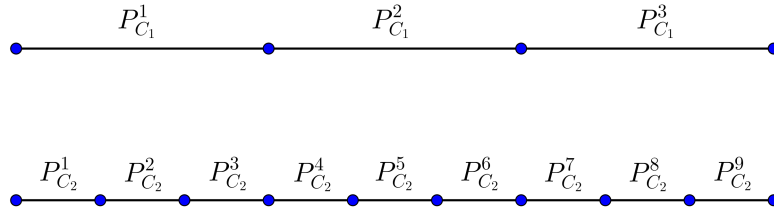


Figure 1: 1st and 2nd Cantor partition sets of $(0, 1)$.

In order to prove Theorem 2.1, let's first prove the following lemmas:

Lemma 2.4 (Average properties of indexes set). *If $s = \frac{1}{2}(m + n)$, then $P_{C_i}^s \subset M_{P_{C_i}^m \cup P_{C_i}^n}$.*

Proof. Without the loss of generality, let's assume that $m < s < n$. Denote x_l, x_r, y_l, y_r to be left and right end points for $P_{C_i}^m$ and $P_{C_i}^n$ respectively. Take any point $z \in P_{C_i}^s$, there exists a $\delta(z)$ such that

$$\max\{z - x_r, y_l - z\} < \delta(z) < \min\{z - x_l, y_r - z\}.$$

And

$$\begin{aligned} x_l &= z - (z - x_l) < z - \delta(z) < z - (z - x_r) = x_r \\ y_l &= z + (y_l - z) < z + \delta(z) < z + (y_r - z) = y_r. \end{aligned}$$

therefore $z - \delta(z) \in P_{C_i}^m$ and $z + \delta(z) \in P_{C_i}^n$, and

$$z = \frac{1}{2}((z - \delta(z)) + (z + \delta(z))) \in M_{P_{C_i}^m \cup P_{C_i}^n}.$$

Hence $P_{C_i}^s \subset M_{P_{C_i}^m \cup P_{C_i}^n}$. □

Lemma 2.5 (Scaling and translation property of the average set). *If $S = c_1T + c_2$ ($c_1 \neq 0$) and $M_T = (a, b)$, then $M_S = c_1(a, b) + c_2$.*

Proof. Pick any point $z_s \in M_S$, there exist $x_s, y_s \in S$ such that $z_s = \frac{1}{2}(x_s + y_s)$. Since $S = c_1T + c_2$, there exist $x_t, y_t \in T$ such that

$$x_s = c_1x_t + c_2, \text{ and } y_s = c_1y_t + c_2.$$

Therefore

$$\begin{aligned} z_s &= \frac{1}{2}(x_s + y_s) \\ &= \frac{1}{2}((c_1x_t + c_2) + (c_1y_t + c_2)) \\ &= c_1 \cdot \frac{1}{2}(x_t + y_t) + c_2 \\ &= c_1z_t + c_2, \end{aligned}$$

where $z_t = \frac{1}{2}(x_t + y_t) \in M_T$. So $z_s \in c_1M_T + c_2$ and then $M_S \subset c_1M_T + c_2$.

On the other hand, pick any point $z_t \in M_T$, there exists a $x_t, y_t \in M_T$ such that $z_t = \frac{1}{2}(x_t + y_t)$. As $S = c_1T + c_2$, $T = \frac{1}{c_1}(S - c_2)$. So there exist $x_s, y_s \in S$ such that

$$x_t = \frac{1}{c_1}(x_s - c_2), \text{ and } y_t = \frac{1}{c_1}(y_s - c_2).$$

Therefore

$$\begin{aligned} z_t &= \frac{1}{2}(x_t + y_t) \\ &= \frac{1}{2} \left(\frac{1}{c_1}(x_s - c_2) + \frac{1}{c_1}(y_s - c_2) \right) \\ &= \frac{1}{c_1} \left(\frac{1}{2}(x_s + y_s) - c_2 \right) \\ &= \frac{1}{c_1}(z_s - c_2), \end{aligned}$$

where $z_s = \frac{1}{2}(x_s + y_s) \in M_S$. So $z_t \in \frac{1}{c_1}(M_S - c_2)$ and then $M_T \subset \frac{1}{c_1}(M_S - c_2)$, i.e. $c_1M_T + c_2 \subset M_S$. Hence $M_S = c_1M_T + c_2$. □

2.2 An analytic argument

Now we will use the lemmas above to prove Theorem 2.1.

Proof. **Claim 1:** $M_{C_1} = (0, 1)$.

Proof of Claim 1: Let $P_{C_1}^1 = (0, \frac{1}{3})$, $P_{C_1}^2 = (\frac{1}{3}, \frac{2}{3})$ and $P_{C_1}^3 = (\frac{2}{3}, 1)$ be the 1st Cantor partition set in Definition 2.3.

- (i) For any $z \in P_{C_1}^1$, since $P_{C_1}^1$ is open, there exists an open interval I_z centered at z such that $I_z \subset P_{C_1}^1$. Therefore, $P_{C_1}^1 \in M_{C_1}$. Similarly, $P_{C_1}^3 \in M_{C_1}$.
- (ii) For any point $z \in P_{C_1}^2$, without the loss of generality, $\frac{1}{3} < z \leq \frac{1}{2}$ (i.e. the other half $\frac{1}{2} \leq z < \frac{2}{3}$ can be solved by symmetry), there exists a $\delta(z)$ satisfying $\frac{1}{3} < \delta(z) < z$ such that

$$\begin{aligned} 0 < z - \delta(z) &< \frac{2}{3} - \delta(z) < \frac{1}{3} \\ \frac{2}{3} < z + \delta(z) &< z + z < 1 \end{aligned}$$

Therefore $z - \delta(z) \in P_{C_1}^1 \subset C_1$ and $z + \delta(z) \in P_{C_1}^3 \subset C_1$, and

$$z = \frac{1}{2}((z + \delta(z)) + (z - \delta(z))) \in M_{C_1}.$$

Hence $P_{C_1}^2 \subset M_{C_1}$.

- (iii) Since

$$\frac{1}{3} = \frac{1}{2} \left(0 + \frac{2}{3} \right), \quad \frac{2}{3} = \left(\frac{1}{3} + 1 \right),$$

and $\{0, \frac{1}{3}, \frac{2}{3}, 1\} \in C_1$, $\{\frac{1}{3}, \frac{2}{3}\} \in M_{C_1}$.

- (iv) There's no way to get 0 or 1 for M_{C_1} .

Finally,

$$M_{C_1} = (\cup_{i=1}^3 P_{C_1}^i) \cup \left\{ \frac{1}{3}, \frac{2}{3} \right\} = (0, 1).$$

Claim 2: $M_{C_i} = (0, 1)$ for all i .

Proof of Claim 2: Let $\{P_{C_i}^k\}$ be the i th Cantor partition set of $[0, 1]$. Assume when $i = n$, $M_{C_n} = (0, 1)$.

- (a) $M_{C_i} \supset (0, \frac{1}{3}) \cup (\frac{1}{3}, \frac{2}{3})$ for all i .
Since $M_{C_n} = (0, 1)$, then $M_{C_n} \supset (0, \frac{1}{3}) \cup (\frac{1}{3}, \frac{2}{3})$. When $i = n + 1$, since

$$C_{n+1} = \frac{1}{3}C_n \cup \left(\frac{2}{3} + \frac{1}{3}C_n \right),$$

and by Lemma 2.5,

$$\begin{aligned} M_{\frac{1}{3}C_n} &= \frac{1}{3}M_{C_n} = \frac{1}{3}(0, 1) = \left(0, \frac{1}{3} \right), \\ M_{\frac{2}{3} + \frac{1}{3}C_n} &= \frac{2}{3} + \frac{1}{3}M_{C_n} = \frac{2}{3} + \frac{1}{3}(0, 1) = \frac{2}{3} + \left(0, \frac{1}{3} \right) = \left(\frac{2}{3}, 1 \right), \end{aligned}$$

$M_{C_{n+1}} \supset (0, \frac{1}{3}) \cup (\frac{2}{3}, 1)$. Hence by induction $M_{C_i} \supset (0, \frac{1}{3}) \cup (\frac{1}{3}, \frac{2}{3})$ for all i .

- (b) $M_{C_i} \supset \{\frac{1}{3}, \frac{2}{3}\}$ for all i .
For each i ,

$$\frac{1}{3} = \frac{1}{2} \left(0 + \frac{2}{3} \right), \quad \frac{2}{3} = \frac{1}{2} \left(\frac{1}{3} + 1 \right),$$

and $\{0, \frac{1}{3}, \frac{2}{3}, 1\} \subset C_i$, therefore $M_{C_i} \supset \{\frac{1}{3}, \frac{2}{3}\}$ for all i .

- (c) $M_{C_i} \supset (\frac{1}{3}, \frac{2}{3})$ for all i .

Since $M_{C_n} = (0, 1)$, then $M_{C_n} \supset (\frac{1}{3}, \frac{2}{3})$. For $i = n + 1$, we will use the assumption $M_{C_n} = (0, 1)$ and the following trick to show $M_{C_{n+1}} \supset (\frac{1}{3}, \frac{2}{3})$.

Take any $z \in (\frac{1}{3}, \frac{2}{3}) = (\frac{3^n}{3^{n+1}}, \frac{2 \cdot 3^n}{3^{n+1}}) = \cup_{k=2 \cdot 3^n+1}^{2 \cdot 3^n} P_{C_{n+1}}^k$, we want to show

$$z = \frac{1}{2}(a + b), \tag{1}$$

where $a \in (\cup_{k=1}^{3^n} P_{C_{n+1}}^k) \cap C_{n+1}$ and $b \in (\cup_{k=2 \cdot 3^n+1}^{3^{n+1}} P_{C_{n+1}}^k) \cap C_{n+1}$. Notice that since $z \in (\frac{1}{3}, \frac{2}{3})$, there exists some $c \in (0, \frac{1}{3})$, such that

$$z = \frac{1}{3} + c = \frac{3^n}{3^{n+1}} + c.$$

Similarly since $C_{n+1} = \frac{1}{3}C_n \cup (\frac{2}{3} + \frac{1}{3}C_n)$, we know

$$\cup_{k=2 \cdot 3^n+1}^{3^{n+1}} P_{C_{n+1}}^k \cap C_{n+1} = \frac{2}{3} + (\cup_{k=1}^{3^n} P_{C_{n+1}}^k) \cap C_{n+1}.$$

So there exists a $d \in \cup_{k=1}^{3^n} P_{C_{n+1}}^k \cap C_{n+1} \subset (0, \frac{1}{3})$. such that $b = \frac{2}{3} + d$. Therefore Eq. (1) is reduced to for any $c \in (0, \frac{1}{3})$, find $a, d \in (\cup_{k=1}^{3^n} P_{C_{n+1}}^k \cap C_{n+1}) \subset (0, \frac{1}{3})$ such that

$$c = \frac{1}{2}(a + d).$$

Note that $\cup_{k=1}^{3^n} P_{C_{n+1}}^k \cap C_{n+1} = (\cup_{k=1}^{3^{n+1}} P_{C_{n+1}}^k \cap C_{n+1}) \cap (0, \frac{1}{3}) = \frac{1}{3}C_n$. And in (a), it is already shown that $M_{\frac{1}{3}C_n} = (0, \frac{1}{3})$. So the existence for a, d are guaranteed. Hence Eq. (1) is true and therefore $M_{C_{n+1}} \supset (\frac{1}{3}, \frac{2}{3})$ is true for all i .

Therefore $M_{C_i} = (0, 1)$ for all i .

Claim 3: $M_{\mathcal{C}} = \bigcap_{i=1}^{\infty} M_{C_i} = (0, 1)$.

- (d) One direction $M_{\mathcal{C}} \subset \bigcap_{i=1}^{\infty} M_{C_i} = (0, 1)$ is trivial, since $\mathcal{C} \subset C_i$ for all i , then $M_{\mathcal{C}} \subset M_{C_i}$ for all i . Hence $M_{\mathcal{C}} \subset \bigcap_{i=1}^{\infty} M_{C_i} = (0, 1)$.
(e) For the other direction, pick any $z \in (0, 1)$, that exists at least one pair $(x_n^k, y_n^k) \in C_n$ such that

$$z = \frac{x_n^k + y_n^k}{2}.$$

Define

$$x_n^0 = \min \left\{ x_n^k : z = \frac{x_n^k + y_n^k}{2}, x_n^k \in C_n \right\}.$$

Claim: $\{x_n^0\}$ converges to x and $x \in \mathcal{C}$.

Proof of claim: First we will show $\{x_n^0\}$ is nondecreasing. Assume it is not true, then there exists an m such that $x_{m+1}^0 \geq x_m^0$. However, $x_{m+1}^0 \in C_{m+1} \subset C_m$, and

$$z = \frac{x_{m+1}^0 + y_{m+1}^0}{2}$$

it contradicts with the construction of x_m^0 . Therefore x_n^0 is nondecreasing. Moreover, $0 \leq x_n^0 < z$, by monotone convergence theorem, $\{x_n^0\}$ converges. Denote

$$x = \lim_{m \rightarrow \infty} x_n^0.$$

For any N , $\{x_n\}_{n \geq N} \subset C_N$. Since C_N is compact, $x_n^0 \rightarrow x \in C_N$. Therefore $x \in \bigcap_{n=1}^{\infty} C_n = \mathcal{C}$. Similarly, $\{y_n^0\}$ is nonincreasing since $y_n^0 = 2z - x_n^0$, and $z < y_n^0 < 1$. Again, by monotone convergence theorem and the same argument, $y_n^0 \rightarrow y \in \mathcal{C}$. Hence $z = \frac{x+y}{2} \in M_C$. Therefore $(0, 1) \subset M_C$.

Finally, let $B = \mathcal{C}$ and $S = M_C$, then for any point $x \in S$, B contains a sphere around x , and $\mathcal{H}^1(S) = 1$ but $\mathcal{H}^1(B) = 0$. In fact, we can have countably many copies of B such that $\mathcal{H}^1(B) = 0$ while $\mathcal{H}^1(S) = \infty$. \square

2.3 A geometric argument

Theorem 2.6. Suppose C , C_i and M_{C_i} are defined as above. Then $M_{C_i} = (0, 1)$ for all integers $i \geq 1$.

Proof. We will show that $M_{C_i} = (0, 1)$ for all integers $i \geq 1$ by inducting on i .

Base Case : Let $i = 1$, and consider the partition set P_i . To show that M_{C_1} contains $P_{C_1}^1$, notice that for any point $m \in P_{C_1}^1$, there exists an $r > 0$ dependent on m , such that $m - r$ and $m + r$ are contained in $P_{C_1}^1$. Similarly, M_{C_1} contains $P_{C_1}^3$.

Now, to show that $P_{C_1}^2 \cup \{\frac{1}{3}, \frac{2}{3}\} \subset M_{C_1}$ we have that for any point $m \in P_{C_1}^2$, $m + \frac{1}{3}$ and $m - \frac{1}{3}$ are both contained in C_1 . Furthermore, $\frac{1}{3} = \frac{1}{2}(0 + \frac{2}{3})$ and $\frac{2}{3} = \frac{1}{2}(\frac{1}{3} + 1)$ in which case we have shown our base case.

Induction : By the inductive hypothesis, suppose the theorem holds for $i = n$ for some $n \in \mathbb{N}$. We will show that the theorem holds true for $i = n + 1$. That is, we will show that $M_{C_{n+1}}$ contains $(0, 1)$.

First consider the scaled version of C_n , i.e. $\frac{1}{3}C_n$. By the scaling property 2.5, since M_{C_n} contains $(0, 1)$, $M_{\frac{1}{3}C_n}$ contains $(0, \frac{1}{3})$ and hence $M_{C_{n+1} \cap (0, \frac{1}{3})} \subset M_{C_{n+1}}$ contains $(0, \frac{1}{3})$. By the same scaling property, we get that that $M_{\frac{1}{3}C_n + \frac{2}{3}}$ contains $(\frac{2}{3}, 1)$ and hence, $M_{C_{n+1}}$ contains $(\frac{2}{3}, 1)$.

Lets take a quick break here to notice that $\frac{1}{3}$ and $\frac{2}{3}$ are both contained in $M_{C_{n+1}}$ because $0, \frac{1}{3}, \frac{2}{3}$, and 1 are all in C_{n+1} just as we showed in the base case. So in the following, we just need to argue that $M_{C_{n+1}}$ contains $(\frac{1}{3}, \frac{2}{3})$.

Consider the partitions in $P_{n+1} \cap (\frac{1}{3}, \frac{2}{3})$ and $P_n \cap (1/3, 2/3)$, which are exactly $\mathcal{P}_{n+1} = \{P_{C_{n+1}}^{3^{n+1}}, P_{C_{n+1}}^{3^n+2}, \dots, P_{C_{n+1}}^{2 \cdot 3^n}\}$, and $\mathcal{P}_n = \{P_{C_n}^{3^{n-1}+1}, \dots, P_{C_n}^{2 \cdot 3^{n-1}}\}$ respectively, see Fig 3.

Pick $\bar{P}_{C_n} \in \mathcal{P}_n$ and partition \bar{P}_{C_n} into thirds to get $\{\bar{P}_{C_n}^1, \bar{P}_{C_n}^2, \bar{P}_{C_n}^3\}$. It is important to notice that $\bar{P}_{C_n}^m \in \mathcal{P}_{n+1}$ for each $m = 1, 2, 3$.

By the inductive hypothesis, there exists distinct partitions $P_{C_n}^k$ and $P_{C_n}^j$ contained in C_n for which the midpoint set of $P_{C_n}^k \cup P_{C_n}^j$ contains \bar{P}_{C_n} .

As we did with \bar{P}_{C_n} , we may partition $P_{C_n}^k$ and $P_{C_n}^j$ into thirds, to get $\{P_{C_n}^{k_1}, P_{C_n}^{k_2}, P_{C_n}^{k_3}\}$ and $\{P_{C_n}^{j_1}, P_{C_n}^{j_2}, P_{C_n}^{j_3}\}$. However, only $\{P_{C_n}^{k_1}, P_{C_n}^{k_3}\}$ and $\{P_{C_n}^{j_1}, P_{C_n}^{j_3}\}$ are in C_{n+1} .

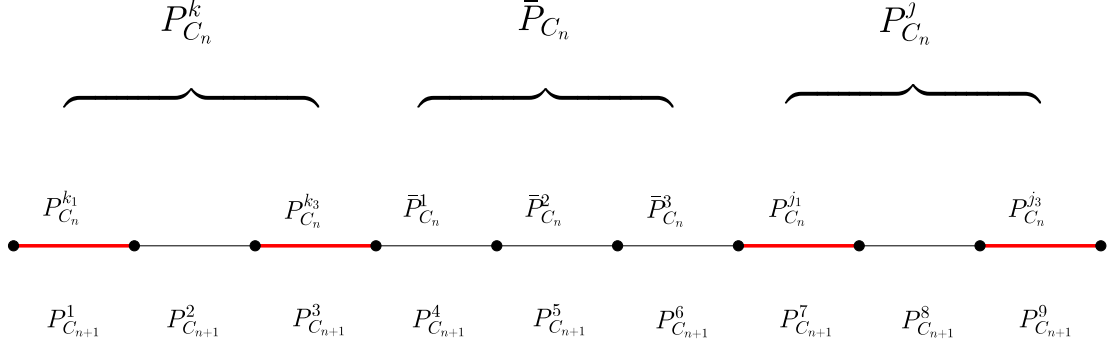


Figure 2: For each \bar{P}_{C_n} , we go to the partitions $P_{C_n}^k$ and $P_{C_n}^j$ for which $M_{P_{C_n}^k \cup P_{C_n}^j}$ contains $[\frac{1}{3}, \frac{2}{3}]$. The first thirds of $P_{C_n}^k$ and $P_{C_n}^j$ gives us $\bar{P}_{C_n}^1$, and the last thirds of $P_{C_n}^k$ and $P_{C_n}^j$ gives us $\bar{P}_{C_n}^3$. To get the middle, $\bar{P}_{C_n}^2$, we use the first and last third of $P_{C_n}^k$ and $P_{C_n}^j$ respectively.

Figure 3: Partition argument

Hence, we get

$$\begin{aligned} M_{P_{C_n}^{k_1} \cup P_{C_n}^{j_1}} &= \bar{P}_{C_n}^1 \\ M_{P_{C_n}^{k_3} \cup P_{C_n}^{j_3}} &= \bar{P}_{C_n}^3 \\ M_{P_{C_n}^{k_1} \cup P_{C_n}^{j_3}} &= \bar{P}_{C_n}^2. \end{aligned} \tag{2}$$

Since this argument holds for arbitrary $\bar{P}_{C_{k-1}}$, we may therefore iterate through \mathcal{P}_{n+1} by iterating through \mathcal{P}_n and showing (2); therefore showing that $[\frac{1}{3}, \frac{2}{3}] \subset M_{C_k}$.

By the principle of mathematical induction we have shown that $(0, 1) \subset M_{C_i}$ for all $i \geq 1$. \square

Theorem 2.7. Suppose \mathcal{C} , C_i and M_{C_i} are defined as above. Then

$$\bigcap_{i=1}^{\infty} M_{C_i} = M_{\mathcal{C}}.$$

Proof. To show $\cap_{i=1}^{\infty} M_{C_i} \subset M_{\mathcal{C}}$, we just need to notice that \mathcal{C} being contained in all C_i , implies $M_{\mathcal{C}}$ is contained in all M_{C_i} .

To show that $\cap_{i=1}^{\infty} M_{C_i} \subset M_{\mathcal{C}}$, let $z \in \cap_{i=1}^{\infty} M_{C_i}$. For each $i = 1, 2, \dots$ there exists $p_i := (x_i, y_i) \in C_i \times C_i$ such that $(x_i + y_i)/2 = z$. Now, for the bounded sequence (p_i) , there exists a subsequence (p_{i_j}) converging to $p := (x, y)$. To show that $p \in \mathcal{C} \times \mathcal{C}$, assume by way of contradiction that p is in $\mathbb{R}^2 \setminus \mathcal{C} \times \mathcal{C}$.

First notice that since $\mathcal{C} \times \mathcal{C}$ is closed, $\mathbb{R}^2 \setminus \mathcal{C} \times \mathcal{C}$ is open, and hence there exists $\epsilon > 0$ for which the open ball $B(p, \epsilon) \subset (\mathbb{R}^2 \setminus \mathcal{C} \times \mathcal{C})$.

Now, since $\mathcal{C} \times \mathcal{C} = \bigcap_{i=1}^{\infty} (C_i \times C_i)$, and since $C_{i+1} \subset (C_i \times C_i)$ for all i , there exists $N_1 \in \mathbb{N}$ such that $(C_i \times C_i) \cap B(p, \epsilon) = \emptyset$ for all $i > N_1$. However, since $p_{i_j} \rightarrow p$ as $j \rightarrow \infty$, there exists $N_2 \in \mathbb{N}$ for which $p_{i_j} \in B(p, \epsilon)$ for all $j > N_2$.

Hence for any $j > \max\{N_1, N_2\}$, we have that p_{i_j} is in both $C_{i_j} \times C_{i_j}$ and $B(p, \epsilon)$; contradicting the fact that $(C_i \times C_i) \cap B(p, \epsilon) = \emptyset$ for all $i > N_1$.

We therefore get that, $p \in \mathcal{C} \times \mathcal{C}$, $(x + y)/2 = z$ and hence $z \in M_{\mathcal{C}}$ thus concluding the proof. \square

By the previous two theorems, we get that the midpoint set of the 1/3-Cantor set is the interval $(0, 1)$. We therefore get that the positive measure set, $S = M_{\mathcal{C}} = (0, 1)$ contains spheres centered about the zero measure set, \mathcal{C} .

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